2. PRODUCTION FUNCTIONS

LEARNING OBJECTIVES

1. How is the output of companies modeled?
2. Are there any convenient functional forms?

The firm transforms inputs into outputs. For example, a bakery takes inputs like flour, water, yeast, labor, and heat and makes loaves of bread. An earth moving company combines capital equipment, ranging from shovels to bulldozers with labor in order to digs holes. A computer manufacturer buys parts "off-the-shelf" like disk drives and memory, with cases and keyboards and combines them with labor to produce computers. Starbucks takes coffee beans, water, some capital equipment, and labor to brew coffee.

Many firms produce several outputs. However, we can view a firm that is producing multiple outputs as employing distinct production processes. Hence, it is useful to begin by considering a firm that produces only one output. We can describe this firm as buying an amount x_1 of the first input, x_2 of the second input, and so on (we'll use xn to denote the last input), and producing a quantity of the output. The production function that describes this process is given by $y = f(x_1, x_2, \dots, xn)$.

The **production function** is the mapping from inputs to an output or outputs.

For the most part we will focus on two inputs in this section, although the analyses with more than inputs is "straightforward."

Example: The **Cobb-Douglas production function** is the product of each input, x, raised to a given power. It takes the form, $f(x_1, x_2, \dots, x_n) = a_0 x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$.

The constants a_1 through an are typically positive numbers less than one. For example, with two goods, capital K and labor L, the Cobb-Douglas function becomes, $a_0 KaLb$. We will use this example frequently. It is illustrated, for $a_0 = 1$, $a = 1/3$, and $b = 2/3$, in Figure 2.1.

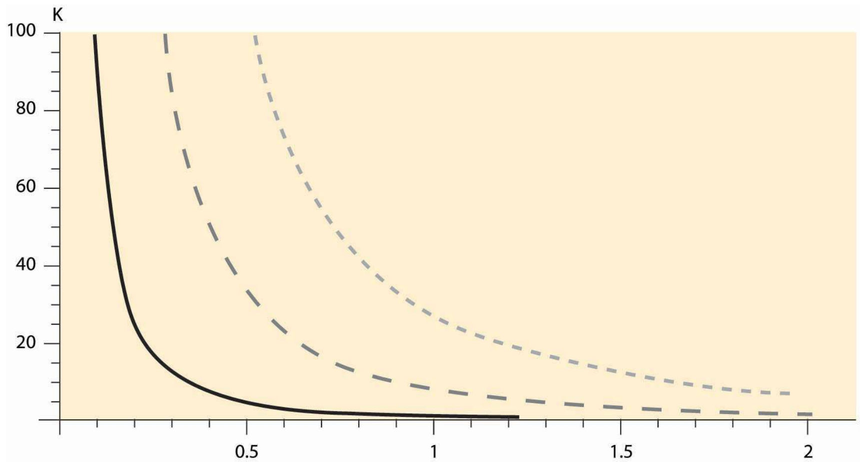
Production function

The mapping from inputs to an output or outputs.

Cobb-Douglas production function

A production function that is the product of each input, x, raised to a given power.

FIGURE 2.1 Cobb-Douglas isoquants

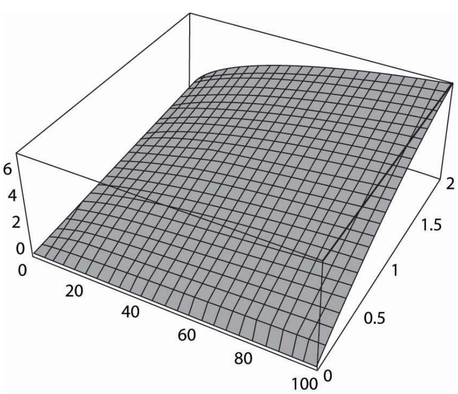


Isoquants

Curves that describe all the combinations of inputs that produce the same level of output.

Figure 2.1 illustrates three isoquants for the Cobb-Douglas production function. An **isoquant**, which means "equal quantity," is a curve that describes all the combinations of inputs that produce the same level of output. In this case, given $a = 1/3$ and $b = 2/3$, we can solve $y = KaLb$ for K to obtain $K = y^3 L^{-2}$. Thus, $K = L^{-2}$ gives the combinations of inputs yielding an output of 1, which is denoted by the dark, solid line in Figure 2.1 The middle, grey dashed line represents an output of 2, and the dotted light-grey line represents an output of 3. Isoquants are familiar contour plots used, for example, to show the height of terrain or temperature on a map. Temperature isoquants are, not surprisingly, called isotherms.

FIGURE 2.2 The production function



Isoquants provide a natural way of looking at production functions and are a bit more useful to examine than 3-D plots like the one provided in Figure 2.2.

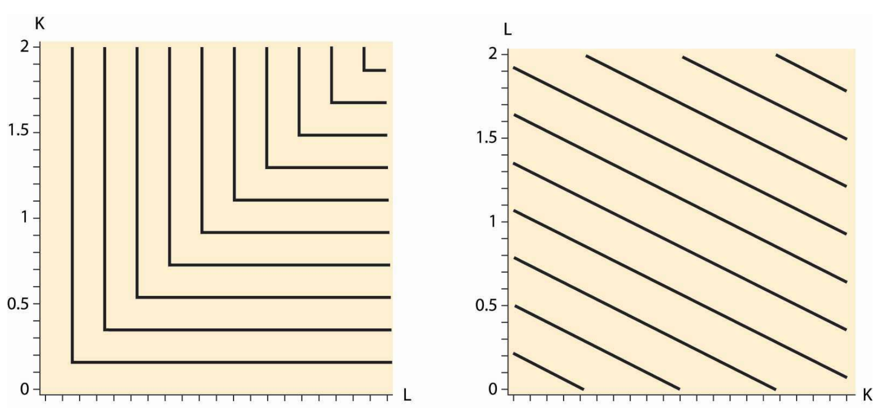
The fixed-proportions production function comes in the form $f(x_1, x_2, ..., x_n) = Min\{a_1 x_1,\ a_2 x_2,\ ...,\ a_n x_n\}$.

Fixed-proportions production function

A production function that requires inputs be used in fixed proportions to produce output.

The **fixed-proportions production function** is a production function that requires inputs be used in fixed proportions to produce output. It has the property that adding more units of one input in isolation does not necessarily increase the quantity produced. For example, the productive value of having more than one shovel per worker is pretty low, so that shovels and diggers are reasonably modeled as producing holes using a fixed-proportions production function. Moreover, without a shovel or other digging implement like a backhoe, a barehanded worker is able to dig so little that he is virtually useless. Ultimately, the size of the holes is determined by min {number of shovels, number of diggers}. Figure 2.3 illustrates the isoquants for fixed proportions. As we will see, fixed proportions make the inputs "perfect complements."

FIGURE 2.3 Fixed-proportions and perfect substitutes



Two inputs K and L are perfect substitutes in a production function f if they enter as a sum; so that, $f(K, L, x3, \dots, xn) = g(K + cL, x3, \dots, xn)$, for a constant c. Another way of thinking of **perfect substitutes** is that they are two goods that can be substituted for each other at a constant rate while maintaining the same output level. With an appropriate scaling of the units of one of the variables, all that matters is the sum of the two variables, not their individual values. In this case, the isoquants are straight lines that are parallel to each other, as illustrated in Figure 2.3.

Perfect substitutes

Two goods that can be substituted for each other at a constant rate while maintaining the same output level.

The **marginal product** of an input is just the derivative of the production function with respect to that input.[3] An important property of marginal product is that it may be affected by the level of other inputs employed. For example, in the Cobb-Douglas case with two inputs[4] and for constant A:

Marginal product

The derivative of the production function with respect to an input.

$$f(K, L) = AK^{\alpha}L^{\beta},$$

the marginal product of capital is

$$\frac{\partial f}{\partial K}(K, L) = \alpha AK^{\alpha - 1}L^{\beta}.$$

If α and β are between zero and one (the usual case), then the marginal product of capital is increasing n the amount of labor, and it is decreasing in the amount of capital employed. For example, an extra computer is very productive when there are many workers and a few computers, but it is not so productive where there are many computers and a few people to operate them.

The **value of the marginal product** of an input is the marginal product times the price of the output. If the value of the marginal product of an input exceeds the cost of that input, it is profitable to use more of the input.

Value of the marginal product

The marginal product times the price of the output.

Some inputs are easier to change than others. It can take five years or more to obtain new passenger aircraft, and four years to build an electricity generation facility or a pulp and paper mill. Very skilled labor such as experienced engineers, animators, and patent attorneys are often hard to find and challenging to hire. It usually requires one to spend three to five years to hire even a small number of academic economists. On the other hand, it is possible to buy shovels, telephones, and computers or to hire a variety of temporary workers rapidly, in a day or two. Moreover, additional hours of work can be obtained from an existing labor force simply by enlisting them to work "overtime," at least on a temporary basis. The amount of water or electricity that a production facility uses can be varied each second. A dishwasher at a restaurant may easily use extra water one evening to wash dishes if required. An employer who starts the morning with a few workers can obtain additional labor for the evening by paying existing workers overtime for their hours of work. It will likely take a few days or more to hire additional waiters and waitresses, and perhaps several days to hire a skilled chef. You can typically buy more ingredients, plates and silverware in one day, whereas arranging for a larger space may take a month or longer.

The fact that some inputs can be varied more rapidly than others leads to the notions of the long run and the short run. In the short run, only some inputs can be adjusted, while in the long run all inputs can be adjusted. Traditionally, economists viewed labor as quickly adjustable, and capital equipment as more difficult to adjust. That is certainly right for airlines—obtaining new aircraft is a very slow process—and for large complex factories, and for relatively low-skilled, and hence substitutable, labor. On the other hand, obtaining workers with unusual skills is a slower process than obtaining warehouse or office space. Generally speaking, the long-run inputs are those that are expensive to adjust quickly, while the short-run factors can be adjusted in a relatively short time frame. What factors belong in which category is dependent upon the context or application under consideration.

KEY TAKEAWAYS

- Firms transform inputs into outputs.
- The functional relationship between inputs and outputs is the production function.
- The Cobb-Douglas production function is the product of the ***inputs*** raised to powers, and comes in the form $f(x_1, x_2, \ldots, x_n) = a_0 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ for positive constants $a_1, \ldots, a_n$.
- An isoquant is a curve or surface that traces out the inputs leaving the output constant.
- The fixed-proportions production function comes in the form
- $f(x_1, x_2, \ldots, x_n) = \min\{a_1 x_1, \ a_2 x_2, \ \ldots, \ a_n x_n\}$.
- Fixed proportions make the inputs "perfect complements."
- Two inputs K and L are perfect substitutes in a production function f if they enter as a sum; that is, $f(K, L, x_3, \ldots, x_n) = g(K + cL, x_3, \ldots, x_n)$, for a constant c.
- The marginal product of an input is just the derivative of the production function with respect to that input. An important aspect of marginal products is that they are affected by the level of other inputs.
- The value of the marginal product of an input is just the marginal product times the price of the output. If the value of the marginal product of an input exceeds the cost of that input, it is profitable to use more of the input.
- Some inputs are more readily changed than others.
- In the short run, only some inputs can be adjusted, while in the long run all inputs can be adjusted.
- Traditionally, economists viewed labor as quickly adjustable, and capital equipment as more difficult to adjust.
- Generally speaking, the long-run inputs are those that are expensive to adjust quickly, while the short-run factors can be adjusted in a relatively short time frame.