4. RISK AVERSION

LEARNING OBJECTIVES

1. How should you evaluate gambles?
2. How is risk priced?

There are many risks in life, even if one doesn't add to these risks by intentionally buying lottery tickets. Gasoline prices go up and down, the demand for people trained in your major fluctuates, and house prices change. How do people value gambles? The starting point for the investigation is the von Neumann-Morgenstern[5] utility function. The idea of a **von Neumann-Morgenstern utility function** for a given person is that, for each possible outcome x, there is a value $v(x)$ assigned by the person, and the average value of v is the value the person assigns to the risky outcome. In other words, the von Neumann-Morgenstern utility function is constructed in such a way that a consumer values gambles as if they were the expected utility

von Neumann-Morgenstern utility function

The value of each outcome, constructed in such a way that a consumer values gambles as if they were the expected utility.

This is a "state of the world" approach, in the sense that each of the outcomes is associated with a state of the world, and the person maximizes the expected value of the various possible states of the world. Value here doesn't mean a money value, but a psychic value or utility.

To illustrate the assumption, consider equal probabilities of winning $100 and winning $200. The expected outcome of this gamble is $150—the average of $100 and $200. However, the expected value of the outcome could be anything between the value of $100 and the value of $200. The von Neumann-Morgenstern utility is ½v($100) + ½$v$($200).

The von Neumann-Morgenstern formulation has certain advantages, including the logic that what matters is the average value of the outcome. On the other hand, in many tests, people behave in ways not consistent with the theory.[6] Nevertheless, the von Neumann approach is the prevailing model of behavior under risk.

To introduce the theory, we will consider only money outcomes, and mostly the case of two money outcomes. The person has a von Neumann-Morgenstern utility function v of these outcomes. If the possible outcomes are $x_1, x_2, \ldots, xn$ and these occur with probability $\pi_1, \pi_2, \ldots, \pi n$ respectively, the consumer's utility is

$$u = \pi_1 v(x_1) + \pi_2 v(x_2) + \ldots + \pi_n v(x_n) = \sum_{i=1}^{n} \pi_i v(x_i).$$

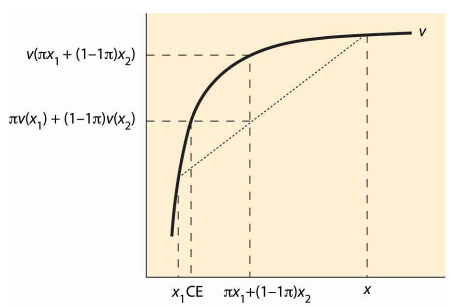
This is the meaning of "having a von Neumann-Morgenstern utility function"—that utility can be written in this weighted sum form.

The first insight that flows from this definition is that an individual dislikes risk if v is concave. To see this, note that the definition of concavity posits that v is concave if, for all π in [0, 1] and all values x_1 and x_2, $v(\pi x_1 + (1-\pi)x_2) \geq \pi v(x_1) + (1-\pi)v(x_2)$.

For smoothly differentiable functions, concavity is equivalent to a second derivative that is not positive. Using induction, the definition of concavity can be generalized to show: $v(\pi_1 x_1 + \pi_2 x_2 + \ldots + \pi_n x_n) \geq \pi_1 v(x_1) + \pi_2 v(x_2) + \ldots + \pi_n v(x_n)$.

That is, a consumer with concave value function prefers the average outcome to the random outcome. This is illustrated in Figure 4.1. There are two possible outcomes, x_1 and x_2. The value x_1 occurs with probability π, and x_2 with probability $1 - \pi$. This means that the average or expected outcome is $\pi x_1 + (1-\pi)x_2$. The value $v\pi\pi x_1 + (1 - \pi)x_2)$ is the value at the expected outcome $\pi x1 + (1-\pi)x2$, while $\pi v(x1) + (1-\pi)v(x2)$ is the average of the value of the outcome. As is plainly visible in the figure, concavity makes the average outcome preferable to the random outcome. People with concave von Neumann-Morgenstern utility functions are known as **risk averse** people—they prefer the expected value of a gamble to the gamble itself.

FIGURE 4.1 Expected utility and certainty equivalents



Risk averse

Preferring the expected value of a gamble to the gamble.

Certainty equivalent

The amount of money that provides equal utility to the random payoff of the gamble.

Risk premium

The difference between the expected payoff and the certainty equivalent.

Arrow-Pratt measure of risk aversion (absolute risk aversion)

A measure of risk aversion computed as the negative of the ratio of the second derivative of utility divided by the first derivative of utility.

Constant absolute risk aversion (CARA)

Situation in which the measure of risk aversion doesn't change with wealth.

Mean variance preferences

Preference that describe people who value risk linearly with the expected return.

A useful concept is the certainty equivalent of a gamble. The **certainty equivalent** is an amount of money that provides equal utility to the random payoff of the gamble. The certainty equivalent is labeled *CE* in the figure. Note that *CE* is less than the expected outcome, if the person is risk averse. This is because risk averse individuals prefer the expected outcome to the risky outcome.

The **risk premium** is defined to be the difference between the expected payoff (this is expressed as $\pi x1 + (1 - \pi)x2$ in the figure) and the certainty equivalent. This is the cost of risk—it is the amount of money an individual would be willing to pay to avoid risk. This means as well that the risk premium is the value of insurance. How does the risk premium of a given gamble change when the base wealth is increased? It can be shown that the risk premium falls as wealth increases for any gamble, if and only if $-\frac{v''(x)}{v'(x)}$ is decreasing.

The measure $\rho(x) = -\frac{v''(x)}{v'(x)}$ is known as the **Arrow-Pratt measure of risk aversion**[7] , and also as the measure of absolute risk aversion. It is a measure of risk aversion computed as the negative of the ratio of the second derivative of utility divided by the first derivative of utility. To get an idea about why this measure matters, consider a quadratic approximation to v. Let μ be the expected value, and let δ^2 be the expected value of $(x - \mu)^2$. Then we can approximate $v(CE)$ two different ways.

$$v(\mu) + v'(\mu)(CE - \mu) \approx v(CE) = E\left\{v(x)\right\} \approx E\left\{v(\mu) + v'(\mu)(x - \mu) + \tfrac{1}{2}v''(\mu)(x - \mu)^2\right\}$$

thus

$$v(\mu) + v'(\mu)(CE - \mu) \approx E\left\{v(\mu) + v'(\mu)(x - \mu) + \tfrac{1}{2}v''(\mu)(x - \mu)^2\right\}$$

Canceling $v(\mu)$ from both sides and noting that the average value of x is μ, so $E(x - \mu) = 0$, we have $v'(\mu)(CE - \mu) \approx \tfrac{1}{2}v''(\mu)\sigma^2$.

Then, dividing by $v'(x)$, $\mu - CE \approx {}^{1}\!/_{2}\frac{v(\mu)}{v'(\mu)}\sigma^2 = {}^{1}\!/_{2}\rho(\mu)\sigma^2$.

That is, the risk premium—the difference between the average outcome and the certainty equivalent—is approximately equal to the Arrow-Pratt measure times half the variance, at least when the variance is small.

The translation of risk into dollars, by way of a risk premium, can be assessed even for large gambles if we are willing to make some technical assumptions. If a utility has **constant absolute risk aversion, or CARA**, the measure of risk aversion doesn't change with wealth; that is $\rho = -\frac{v''(x)}{v'(x)}$ is a constant. This turns out to imply, after setting the utility of zero to zero, that $v(x) = \frac{1}{\rho}(1 - e^{-\rho x})$.

(This formulation is derived by setting $v(0) = 0$, handling the case of $\rho = 0$ with appropriate limits.) Now also assume that the gamble x is normally distributed with mean μ and variance δ^2. Then the expected value of $v(x)$ is $Ev(x) = \frac{1}{\rho}\left(1 - e^{-\rho\left(\mu - \frac{\rho}{2}\sigma^2\right)}\right)$.

It is an immediate result from this formula that the certainty equivalent, with CARA preferences and normal risks, is $\mu - \frac{\rho}{2}\sigma^2$. Hence, the risk premium of a normal distribution for a CARA individual is $\frac{\rho}{2}\sigma^2$. This formulation will appear when we consider agency theory and the challenges of motivating a risk averse employee when outcomes have a substantial random component.

An important aspect of CARA with normally distributed risks is that the preferences of the consumer are linear in the mean of the gamble and the variance. In fact, given a choice of gambles, the consumer selects the one with the highest value of $\mu - \frac{\rho}{2}\sigma^2$. Such preferences are often called **mean variance preferences**, and they describe people who value risk linearly with the expected return. Such preferences comprise the foundation of modern finance theory.

KEY TAKEAWAYS

- The von Neumann-Morgenstern utility function for a given person is a value $v(x)$ for each possible outcome x, and the average value of v is the value the person assigns to the risky outcome. Under this theory, people value risk at the expected utility of the risk.
- The von Neumann approach is the prevailing model of behavior under risk, although there are numerous experiment-based criticisms of the theory.
- An individual dislikes risk if v is concave.
- For smoothly differentiable functions, concavity is equivalent to a second derivative that is not positive.
- People with concave von Neumann-Morgenstern utility functions are known as risk averse people.
- The certainty equivalent of a gamble is an amount of money that provides equal utility to the random payoff of the gamble. The certainty equivalent is less than the expected outcome if the person is risk averse.
- The risk premium is defined to be the difference between the expected payoff and the certainty equivalent.
- The risk premium falls as wealth increases for any gamble, if and only if $-\frac{v''(x)}{v'(x)}$ is decreasing.
- The measure $\rho(x) = -\frac{v''(x)}{v'(x)}$ is known as the Arrow-Pratt[8] measure of risk aversion, and also as the measure of absolute risk aversion.
- The risk premium is approximately equal to the Arrow-Pratt measure times half the variance when the variance is small.
- Constant absolute risk aversion provides a basis for “mean variance preferences,” the foundation of modern finance theory.