3. PROFIT MAXIMIZATION

LEARNING OBJECTIVE

1. **If firms maximize profits, how will they behave?**

Consider an entrepreneur who would like to maximize profit, perhaps by running a delivery service. The entrepreneur uses two inputs, capital K (e.g., trucks) and labor L (e.g., drivers), and rents the capital at cost r per dollar of capital. The wage rate for drivers is w. The production function is $F(K, L)$; that is, given inputs K and L, the output is $F(K, L)$. Suppose p is the price of the output. This gives a profit of:[5]

$$\pi = pF(K, L) - rK - wL.$$

First, consider the case of a fixed level of K. The entrepreneur chooses L to maximize profit. The value L^* of L that maximizes the function π must satisfy:

$$0 = \frac{\partial \pi}{\partial L} = p\frac{\partial F}{\partial L}(K, L^*) - w.$$

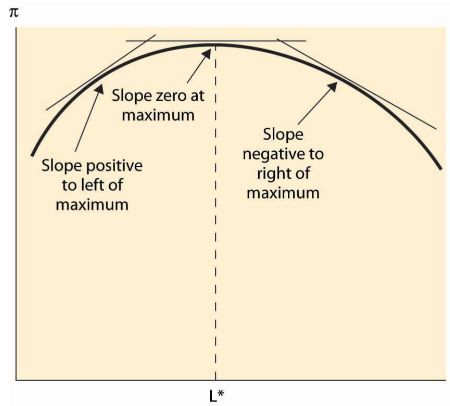
First-order condition

A mathematical condition for optimization stating that the first derivative is zero.

This expression is known as a **first-order condition**, a mathematical condition for optimization stating that the first derivative is zero.[6] The first-order condition recommends that we add workers to the production process up to the point where the last worker's marginal product is equal to his wage (or cost).

In addition, a second characteristic of a maximum is that the second derivative is negative (or nonpositive). This arises because, at a maximum, the slope goes from positive (since the function is increasing up to the maximum) to zero (at the maximum) to being negative (because the function is falling as the variable rises past the maximum). This means that the derivative is falling; or that the second derivative is negative. This logic is illustrated in Figure 3.1.

FIGURE 3.1 Profit-maximizing labor input



The second property is known as the **second-order condition**, a mathematical condition for maximization stating that the second derivative is nonpositive.[7] It is expressed as:

$$0 \geq \frac{\partial^2 \pi}{(\partial L)^2} = p\frac{\partial^2 F}{(\partial L)^2}(K, L^*).$$

Second-order condition

A mathematical condition for maximization stating that the second derivative is nonpositive.

This is enough of a mathematical treatment to establish comparative statics on the demand for labor. Here, we treat the choice L^* as a function of another parameter—the price p, the wage w, or the level of capital K. For example, to find the effect of the wage on the labor demanded by the entrepreneur, we can write:

$$0 = p\frac{\partial F}{\partial L}(K, L^*(w)) - w.$$

This expression recognizes that the choice L^* that the entrepreneur makes satisfies the first-order condition, and results in a value that depends on w. But how does it depend on w? We can differentiate this expression to obtain:

$$0 = p\frac{\partial^2 F}{(\partial L)^2}(K, L^*(w))L^{*\prime}(w) - 1,$$

or

$$L^{*\prime}(w) = \frac{1}{p\frac{\partial^2 F}{(\partial L)^2}(K, L^*(w))} \leq 0.$$

The second-order condition enables one to sign the derivative. This form of argument assumes that the choice L^* is differentiable, which is not necessarily true.

Digression: In fact, there is a form of argument that makes the point without calculus and makes it substantially more general. Suppose $w_1 < w_2$ are two wage levels, and that the entrepreneur chooses L_1 when the wage is w_1 and L_2 when the wage is w_2. Then profit maximization requires that these choices are optimal. In particular, when the wage is w_1, the entrepreneur earns higher profit with L_1 than with L_2:

$$pf(K, L_1) - rK - w_1L_1 \geq pf(K, L_2) - rK - w_1L_2$$

When the wage is w_2, the entrepreneur earns higher profit with L_2 than with L_1:

$$pf(K, L_2) - rK - w_2L_2 \geq pf(K, L_1) - rK - w_2L_1.$$

The sum of the left-hand sides of these two expressions is at least as large as the sum of the right-hand side of the two expressions:

$$pf(K, L_1) - rK - w_1L_1 + pf(K, L_2) - rK - w_2L_2 \geq$$

$$pf(K, L_1) - rK - w_2L_1 + pf(K, L_2) - rK - w_1L_2$$

A large number of terms cancel to yield

$$-w_1L_1 - w_2L_2 \geq -w_2L_1 - w_1L_2$$

This expression can be rearranged to yield

$$(w_1 - w_2)(L_2 - L_1) \geq 0.$$

Revealed preference

Kind that states that choice implies preference.

This shows that the higher labor choice must be associated with the lower wage. This kind of argument, sometimes known as a **revealed preference** kind of argument, states that choice implies preference. It is called "revealed preference" because choices by consumers were the first place the type of argument was applied. It can be very powerful and general, because issues of differentiability are avoided. However, we will use the more standard differentiability-type argument, because such arguments are usually more readily constructed.

The effect of an increase in the capital level K on the choice by the entrepreneur can be calculated by considering L^* as a function of the capital level K:

$$0 = p\frac{\partial F}{\partial L}(K, L^*(K)) - w.$$

Differentiating this expression with respect to K, we obtain

$$0 = p\frac{\partial^2 F}{\partial K \partial L}(K, L^*(K)) + p\frac{\partial^2 F}{(\partial L)^2}(K, L^*(K))L^{*\prime}(K),$$

or

$$L^{*\prime}(K) = \frac{-\frac{\partial^2 F}{\partial K \partial L}(K, L^*(K))}{\frac{\partial^2 F}{(\partial L)^2}(K, L^*(K))}.$$

We know the denominator of this expression is not positive, thanks to the second-order condition, so the unknown part is the numerator. We then obtain the conclusion that:

An increase in capital increases the labor demanded by the entrepreneur if $\frac{\partial^2 F}{\partial K \partial L}(K, L^*(K)) > 0$, and decreases the labor demanded if $\frac{\partial^2 F}{\partial K \partial L}(K, L^*(K)) < 0$.

This conclusion looks like gobbledygook but is actually quite intuitive. Note that $\frac{\partial^2 F}{\partial K \partial L}(K, L^*(K)) > 0$ means that an increase in capital increases the derivative of output with respect to labor; that is, an increase in capital increases the marginal product of labor. But this is, in fact, the definition of a complement! That is, $\frac{\partial^2 F}{\partial K \partial L}(K, L^*(K)) > 0$ means that labor and capital are complements in production—an increase in capital increases the marginal productivity of labor. Thus an increase in capital will increase the demand for labor when labor and capital are complements, and it will decrease the demand for labor when labor and capital are substitutes.

This is an important conclusion because different kinds of capital may be complements or substitutes for labor. Are computers complements or substitutes for labor? Some economists consider that computers are complements to highly skilled workers, increasing the marginal value of the most skilled, but substitutes for lower-skilled workers. In academia, the ratio of secretaries to professors has fallen dramatically since the 1970s as more and more professors are using machines to perform secretarial functions. Computers have increased the marginal product of professors and reduced the marginal product of secretaries, so the number of professors rose and the number of secretaries fell.

The revealed preference version of the effect of an increase in capital is to posit two capital levels, K_1 and K_2, with associated profit-maximizing choices L_1 and L_2. The choices require, for profit maximization, that

$$pF(K_1, L_1) - rK_1 - wL_1 \geq pF(K_1, L_2) - rK_1 - wL_2$$

and

$$pF(K_2, L_2) - rK_2 - wL_2 \geq pF(K_2, L_1) - rK_2 - wL_1.$$

Again, adding the left-hand sides together produces a result at least as large as the sum of the right-hand sides:

$$pF(K_1, L_1) - rK_1 - wL_1 + pF(K_2, L_2) - rK_2 - wL_2 \geq$$
$$pF(K_2, L_1) - rK_2 - wL_1 + pF(K_1, L_2) - rK_1 - wL_2.$$

Eliminating redundant terms yields

$$pF(K_1, L_1) + pF(K_2, L_2) \geq pF(K_2, L_1) + pF(K_1, L_2),$$

or

$$F(K_2, L_2) - F(K_1, L_2) \geq F(K_2, L_1) - F(K_1, L_1)$$

or

$$\int_{K_1}^{K_2} \frac{\partial F}{\partial K}(x, L_2)dx \geq \int_{K_1}^{K_2} \frac{\partial F}{\partial K}(x, L_1)dx,$$ [8]

or

$$\int_{K_1}^{K_2} \frac{\partial F}{\partial K}(x, L_2) - \frac{\partial F}{\partial K}(x, L_1)dx \geq 0,$$

and finally

$$\int_{K_1}^{K_2}\int_{L_1}^{L_2} \frac{\partial^2 F}{\partial K \partial L}(x, y)\, dydx \geq 0$$

Thus, if $K_2 > K_1$ and $\frac{\partial^2 F}{\partial K \partial L}(K, L) > 0$ for all K and L, then $L_2 \geq L_1$; that is, with complementary inputs, an increase in one input increases the optimal choice of the second input. In contrast, with substitutes, an increase in one input decreases the other input. While we still used differentiability of the production function to carry out the revealed preference argument, we did not need to establish that the choice L^* was differentiable to perform the analysis.

Example (Labor demand with the Cobb-Douglas production function): The Cobb-Douglas production function has the form $F(K, L) = AK^{\alpha}L^{\beta}$, for constants A, α, and β, all positive. It is necessary for $\beta < 1$ for the solution to be finite and well defined. The demand for labor satisfies

$$0 = p\frac{\partial F}{\partial L}(K, L^*(K)) - w = p\beta AK^{\alpha}L^{*\beta - 1} - w,$$

or

$$L^* = \left(\frac{p\beta AK^{\alpha}}{w}\right)^{1 / 1 - \beta}.$$

When $\alpha + \beta = 1$, L is linear in capital. Cobb-Douglas production is necessarily complementary; that is, an increase in capital increases labor demanded by the entrepreneur.

KEY TAKEAWAYS

- Profit maximization arises when the derivative of the profit function with respect to an input is zero. This property is known as a first-order condition.
- Profit maximization arises with regards to an input when the value of the marginal product is equal to the input cost.
- A second characteristic of a maximum is that the second derivative is negative (or nonpositive). This property is known as the second-order condition.
- Differentiating the first-order condition permits one to calculate the effect of a change in the wage on the amount of labor hired.
- Revealed preference arguments permit one to calculate comparative statics without using calculus, under more general assumptions.
- An increase in capital will increase the demand for labor when labor and capital are complements, and it will decrease the demand for labor when labor and capital are substitutes.
- Cobb-Douglas production functions are necessarily complements; hence any one input increases as the other inputs increase.